Math 2250 Lab #3: Landing on Target

1. Introduction to the lab program.

Here are some general notes and ideas which will help you with the lab. The purpose of the lab program is to expose you to problems which are more complicated and open-ended than traditional calculus exercises. They will require you to really understand some applications of calculus, in particular learning much more about projectile motion than you would learn in a standard calculus class. We are doing this because learning to solve these kinds of problems is excellent preparation for the science and engineering coursework you have ahead of you and for the career you'll have once you leave college. The students who have take the labs before you asked me to add some general insights about the labs for you to read before you start. These will make even more sense once you've done the lab, but they are good pointers to get ready for the work ahead.

- Don't give up too easily (and don't be afraid of complexity)!

  Every problem that someone will pay you to solve (regardless of your major or career path) will be complicated, messy, and hard to define. All of the easy problems are now done by computers and robots. So now is the time to start learning to deal with complicated problems. This is very different from learning to do well on standardized tests of "mathematical knowledge" or "mathematical ability". The point of the lab is not to make you better at filling in the correct circles on scantron sheets– nobody hires professional circle-fillers!– it's to start the process of making you employable.

- Read the instructions (more than once)!

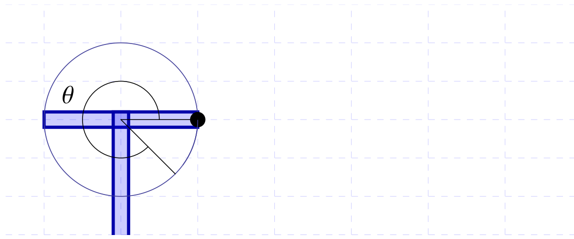
  The labs are a different kind of text than you may have seen before. You may have to read the instructions several times before you understand the question. This is something that you should expect. It often helps to read the instructions aloud.

- Talk with your classmates (even if none of you know the answer)!

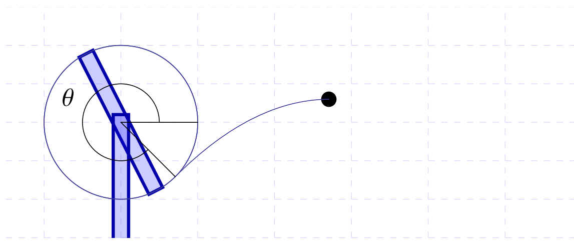
  It is easy to think about talking as a process where knowledge is transmitted from someone who has it to someone who doesn't. This is often true. But talking can also be a way for a group of people to cooperatively generate new knowledge that nobody in the conversation had when the conversation started. In these conversations, you don't have to be the most advanced student in the group to make an important contribution– sometimes asking the right question can be more important than having all the answers. Learning how to have these kinds of conversations is one of the most important things that you can learn from this class.

2. Client Requirements

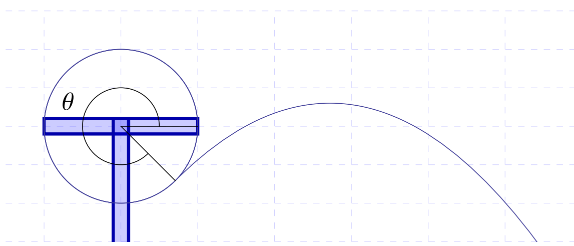
Our client is designing a robot to throw a ball across the table into a cup. The robot will have a rotating arm of length $r\,\mathrm{cm}$ mounted on a stand at height $h\,\mathrm{cm}$ above the table surface. The arm will spin at $k\,\mathrm{rpm}$ and release the ball at angle θ_0 (radians).The pictures below show the basic setup:



The ball at start



after launch



at landing.

The client has now chosen the values $r = 11\mathrm{cm}$ and $h = 15\mathrm{cm}$ for the final design. Our assignment is to come up with a procedure for specifying θ and k in order to hit a target at a fixed distance $\mathrm{TD}_0 \pm 0.5\,\mathrm{cm}$ from the base of the launcher. We will have to explain to client how to solve for θ and k in order to hit the target, and illustrate our method with some particular examples.

3. Implementing the Landing Position Function on a Computer

1. In lab 2, you found the landing position of the ball as a function of the variables r, h, θ, and k. For this lab, you will need to be able to calculate this landing position rapidly for various different values of these variables. There are several ways to do this:
   - use a programming language, such as Java, MATLAB, or Mathematica,
   - use a spreadsheet such as Excel or Google Sheets,
   - program a graphing calculator,
   - use WolframAlpha.

Choose whichever method above that you are most comfortable with, and set up a computation for the landing function. Be sure to explain how your calculation works.

2. Invent *three* combinations of r, h, k, and θ where you can guess the answer, explain your reasoning, and verify that your computer method produces the answer you expected. Then find another student, choose three arbitrary combinations of these variables, and verify that you both get the same answer from your computational methods. Be sure that you vary r and h as well as k and θ.
3. Check the results from your previous computations in Lab 2, and make sure you get the same answers.

4. The Bisection Method

We are now going to use the bisection method to solve for combinations of release angle θ and rotation speed k which land the projectile on target. This requires that you understand the bisection method and can apply it in an organized way.

1. Explain the bisection method. How does it work? What is it for? For extra credit: Can you implement the bisection method on your computing platform?
2. Suppose we fix k, r, and h, so that the landing position function is a function $\mathrm{LP}(\theta)$ of the single variable θ. If $k = 150\,\mathrm{rpm}, r = 11\,\mathrm{cm}, h = 15\,\mathrm{cm}$, use the bisection method to find an interval of θ values $[\theta_a, \theta_b]$ so that $\mathrm{LP}(\theta_a)$ and $\mathrm{LP}(\theta_b)$ are both within $\pm 0.5\,\mathrm{cm}$ of $\mathrm{TD}_0 = 35\,\mathrm{cm}$.
3. The angle sensor on the robot registers angle in 4 degree increments. In our example, where $k = 150\,\mathrm{rpm}, r = 11\,\mathrm{cm}, h = 15\,\mathrm{cm}$, and $\mathrm{TD}_0 = 35\,\mathrm{cm}$, decide whether this angular resolution is good enough to land the ball within $\pm 0.5\,\mathrm{cm}$ of TD_0.
4. Suppose that we fix θ, r, and h, so that the landing position function is a function $\mathrm{LP}(k)$ of the single variable k instead. If $\theta = 290\,\mathrm{deg}, r = 11\,\mathrm{cm}, h = 15\,\mathrm{cm}$, use the bisection method to find an interval of k value $[k_a, k_b]$ so that $\mathrm{LP}(k_a)$ and $\mathrm{LP}(k_b)$ are within $0.5\mathrm{cm}$ of $\mathrm{TD}_0 = 35\mathrm{cm}$.
5. The combination of sensor measurements that the robot uses to measure rotation speed are together accurate to $\pm 5rpm$. Is this rotational speed resolution good enough to land the ball within $\pm 0.5\,\mathrm{cm}$ of TD_0?

5. Sensitivity Analysis

We can see that we can either fix rotation speed and solve for a release angle or fix a release angle and solve for a rotation speed. In both cases, it seems that the robot is not accurate enough in its measurements of speed and angle to land the projectile within the acceptable range of landing positions. How accurate would the robot need to be? This is an error analysis problem, for which we need the theory of linear approximation.

Remember that for any differentiable function $f(x)$ and any value x_0, if we vary x_0 by some small amount Δx, we can estimate the change in the function values by

$$\Delta f \simeq f'(x_0)\Delta x \quad \text{or} \quad |f(x_0+\Delta x) - f(x_0)| \simeq f'(x_0)\Delta x.$$

Let's apply this insight to our functions for landing position $\mathrm{LP}(\theta)$ and $\mathrm{LP}(k)$.

1. Find an expression for the derivative $\frac{d}{d\theta}\,\mathrm{LP}$. This is relatively easy using *Mathematica* or Wolfram Alpha, harder by hand. Try to break the problem down into manageable pieces. If you have

   $$\mathrm{LP}(\theta) = \mathrm{VX}(\theta)\tau(\theta) + \mathrm{PX}(\theta),$$

   where τ is flight time then

   $$\mathrm{LP}'(\theta) = \mathrm{VX}'(\theta)\tau(\theta) + \mathrm{VX}(\theta)\tau'(\theta) + \mathrm{PX}'(\theta)$$

   and you can work out the derivatives $\mathrm{VX}'(\theta)$, $\mathrm{PX}'(\theta)$, and $\tau'(\theta)$ separately.
2. Implement your calculation of the derivative on the same computing platform that you used for LP in part 1.
3. Above, you found an interval $[\theta_a, \theta_b]$ of angles which land the projectile within $0.5\,\mathrm{cm}$ of $35\,\mathrm{cm}$ with $k = 150\,\mathrm{rpm}, r = 11\,\mathrm{cm}, h = 15\,\mathrm{cm}$. Let's use the midpoint of that interval, $(\theta_a + \theta_b)/2$ as our best guess for a θ_0 which lands the projectile as close as we can to the target at $35\,\mathrm{cm}$. Compute the derivative $\mathrm{LP}'(\theta_0)$ and estimate a range of θ values $\Delta\theta$ which makes $\Delta\,\mathrm{LP} = 0.5\,\mathrm{cm}$.
4. Use trial and error to find the actual range of θ values which land the projectile between $34.5\,\mathrm{cm}$ and $35.5\,\mathrm{cm}$. Compare this to the $\Delta\theta$ above (hint: they should be very close).
5. Convert your $\Delta\theta$ values into degrees and into percentage of a full revolution to get an intuitive sense of how accurate the angle sensor needs to be. (In principle, you could have intuition about radians. But very few people actually *do*.) Is this a large range of angles or a small one?
6. Suppose that $\mathrm{LP}'(\theta)$ was larger or smaller. Would the size $\Delta\theta$ of the range of acceptable angles get larger or smaller? Why? What value of $\mathrm{LP}'(\theta)$ would give us the largest $\Delta\theta$? Why?

6. Graphing the problem

We are now going to try to understand the analysis above graphically.

1. Make a graph of the landing position function $\mathrm{LP}(\theta)$ for values of θ between 0 and 2π using the previous values of $k = 150\,\mathrm{rpm}$, $r = 11\,\mathrm{cm}$ and $h = 15\,\mathrm{cm}$.
2. On the graph, locate θ_0 and the endpoints of the interval $\theta_0 \pm \Delta\theta$ of angles with LP values in $35\,\mathrm{cm} \pm 0.5$cm.
3. Looking at the graph, find a new θ_1 where $\mathrm{LP}'(\theta_1)$ is smaller. Plot the endpoints of the interval $\theta_1 \pm \Delta\theta$ with LP values in $LP(\theta_1)\,\mathrm{cm} \pm 0.5$cm.

7. Conclusion

What you've done so far should suggest a conclusion about the best angle to release at for any given rotation speed k. Of course, it doesn't tell you how to find k to finish the job! Note that there are two things you can do at this point: fix k and vary θ (using the bisection method) and fix θ and vary k (using the bisection method).

1. State clearly a procedure for finding the best k and θ for a given landing position. Explain the connection between the graphs and the sensitivity analysis you did above.
2. Find the values of θ and k which have the largest range of θ which lands the projectile within $\pm 0.5\,\mathrm{cm}$ of $35\,\mathrm{cm}$.