Math 2250 Lab #1 : Tennis Ball Missile Defense

1. Introduction to the lab program.

Here are some general notes and ideas which will help you with the lab. The purpose of the lab program is to expose you to problems which are more complicated and open-ended than traditional calculus exercises. They will require you to really understand some applications of calculus, in particular learning much more about projectile motion than you would learn in a standard calculus class. We are doing this because learning to solve these kinds of problems is excellent preparation for the science and engineering coursework you have ahead of you and for the career you'll have once you leave college. The students who have take the labs before you asked me to add some general insights about the labs for you to read before you start. These will make even more sense once you've done the lab, but they are good pointers to get ready for the work ahead.

- Don't give up too easily (and don't be afraid of complexity)!

  Every problem that someone will pay you to solve (regardless of your major or career path) will be complicated, messy, and hard to define. All of the easy problems are now done by computers and robots. So now is the time to start learning to deal with complicated problems. This is very different from learning to do well on standardized tests of "mathematical knowledge" or "mathematical ability". The point of the lab is not to make you better at filling in the correct circles on scantron sheets– nobody hires professional circle-fillers!– it's to start the process of making you employable.

- Read the instructions (more than once)!

  The labs are a different kind of text than you may have seen before. You may have to read the instructions several times before you understand the question. This is something that you should expect. It often helps to read the instructions aloud.

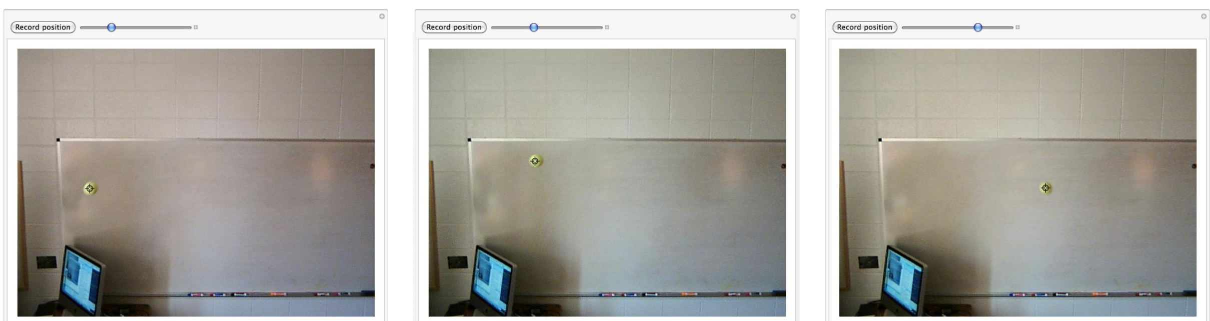
- Talk with your classmates (even if none of you know the answer)!

  It is easy to think about talking as a process where knowledge is transmitted from someone who has it to someone who doesn't. This is often true. But talking can also be a way for a group of people to cooperatively generate new knowledge that nobody in the conversation had when the conversation started. In these conversations, you don't have to be the most advanced student in the group to make an important contribution– sometimes asking the right question can be more important than having all the answers. Learning how to have these kinds of conversations is one of the most important things that you can learn from this class.

2. Client Requirements

In the labs, we'll be playing the role of mathematical consultants to a client organization which needs to solve some practical problems. In this lab, our client has trouble with a neighbor who is constantly throwing tennis balls at their head. The client has designed and built an air-powered, computer controlled tennis ball cannon, and set up a webcam in the driveway together with tracking software that can detect the position of incoming tennis balls. The client needs your help to program the tennis ball cannon to fire a second ball which intercepts and deflects the incoming tennis balls before they reach the client's head. The tennis ball cannon is mounted at a fixed position (x, y) and launches the ball in a fixed direction θ radians from the horizontal at a fixed velocity of V pixels per frame. None of these parameters can be changed while an incoming tennis ball is in flight. However, the computer can fire the tennis ball cannon at any given time. Your ultimate goal in this lab is to explain to the client how to decide when to fire the intercepting ball.

We've gathered data from a test video of a tennis ball thrown by your professor[1]. You will use this data to illustrate your *general* method for solving these problems.



We took measurements of ball position at several times during the flight.

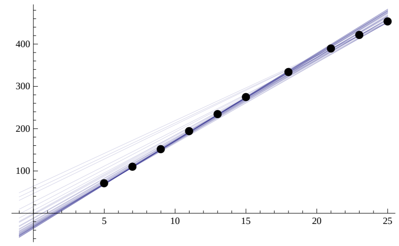
time t	5	7	9	11	13	15	18	21	23	25
x-position $px(t)$	69.5	109	150.5	192	233	273.3	332.5	388	420.5	451.5
y-position $py(t)$	147.6	204.6	249.2	278.6	294.2	294	264.6	206.6	152.6	89.6

3. Predicting the Trajectory of an Incoming Ball from Position Data

We know that the acceleration in the x-direction is zero and that the acceleration in the y direction (due to gravity) is constant. This means that the x data should be described by a linear function and the y data should be described by a quadratic function. Notice that the units here are pixels per frame, so there is no reason to believe that the acceleration due to gravity has a numerical value equal to -9.8, so unlike our class problems, we won't immediately know the coefficient g in the quadratic function for y.

[1]Your professor has insisted that they do *not* throw tennis balls at their neighbors' heads on a regular basis.

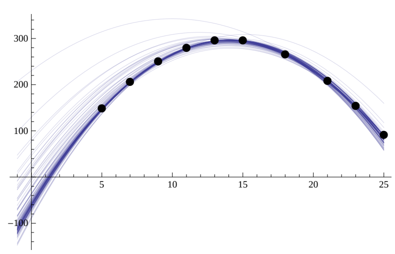
1. Fit a linear function $\mathrm{px}(t) = mt + b$ to the x data above. Here are some things to think about: first, any two points determine a line (you know how to compute the slope of a line through two given points, and you can write a line through one of those points with that slope in point-slope form). But you will need to consider the effect of measurement error on this data as it is only approximately linear. The diagram below shows all of the various lines through two of these data points:



   (1) Give a general procedure for finding a single line which approximately fits all of the data.
   (2) Test your prediction for the data above at every time value and discuss the maximum and average error. The ball is about 10 pixels across. Is your line a good enough prediction for an interception? (If not, revise your procedure and try again.) Please include your prediction and error for every t in the lab writeup.

   You may know how to "do a linear regression" from statistics, or by pressing the magic button on your calculator. I *don't* want you to do this here, but rather to invent your own method which you can fully explain. The reason for this is that the mathematics of a linear regression is quite complicated (it's beyond the level of this course), and I'd much rather see you practice thinking about a problem than see you recite directions for a procedure that you don't fully understand.

2. We now need to find a quadratic function $\mathrm{py}(t) = at^2 + bt + c$ which matches given y data and test our procedure on the y-data above.
   (1) Suppose we have a general quadratic equation $\mathrm{py}(t) = at^2 + bt + c$ and three data points (t_1, y_1), (t_2, y_2), and (t_3, y_3) on the graph of the function $\mathrm{py}(t)$ Show that we can set up a system of linear equations for the coefficients a, b, and c in terms of these data points. In general, this system has a unique solution, so "three points determine a quadratic" just as "two points determine a line".
   (2) The picture below shows all of the quadratic functions determined by triples of y data points.



   Come up with a procedure to arrive at a single parabola representing all of a given set of data.
   (3) Test your plan by carrying it out for the y data above and checking the error at each t value. Is your parabola a good enough prediction for an interception? (If not, revise the procedure and try again.)

Again, you may know how to do a "quadratic regression" using a computer. For the same reasons as above, I don't want you to do this here. Many of you will eventually come up with a method which involves solving a system of equations. If you do, I'd like to see the explicit solution of your system in the lab writeup. I don't want you to use Cramer's rule (for example), because it's another magical procedure where you don't really understand why it works. Those can be very dangerous because they are very hard to check.

3. Summarize your procedures for finding the trajectory of an incoming ball for the client. Your directions should enable the client to start with a table of data such as the one at the start of this section and end with a predicted trajectory for a tennis ball. Use the trajectory you found using this data as an example to illustrate your method. However, your writeup should be more than your computation of a single trajectory from the data set above; it should be a general method which would apply to any data set provided by the client.
4. (Bonus Credit) In part 2 of this problem, you actually found an approximate value for the gravitational constant in pixels per frame. Given that there are 30 frames in 1 second, use your knowledge of g in the meters/seconds unit system to solve for the number of pixels per meter in the test video[2]. Using this information, compute the initial velocity of this particular tennis ball in meters per second and miles per hour.

4. Predicting the Trajectory of the Intercepting Ball from Cannon Data

Suppose that we launch an intercepting tennis ball with velocity V pixels per frame from position LX, LY (launcher-X and launcher-Y) at time 0, and that the cannon is fixed to launch at an angle of θ radians from the horizontal (so that at launch, the *horizontal* or x velocity of the ball is $V\cos\theta$ and the *vertical* or y velocity of the ball is $V\sin\theta$). We're going to use capital letters for the equations and constants for the intercepting ball in order to distinguish them from the equations for the original ball, but don't be confused– both balls are following the equations for projectile motion that we learned in class.

1. Find the linear function $\mathrm{PX}(t)$ for the x position of the intercepting ball using the equations of projectile motion assuming that we launch at time 0. Your answer should be a function of the variables V, θ, LX and LY. (We can't give numerical values to these variables without knowing where the launcher for the intercepting ball will be located and aimed, and how fast it launches the intercepting ball.)
2. Find the quadratic function $\mathrm{PY}(t)$ for the y position of the intercepting ball assuming that we launch at time 0. Write an explanation for your client of why the coefficient of t^2 in this function is not $-9.8/2$. Again, your answer should be a function of the variables V, θ, LX and LY.
3. Summarize your procedure for finding the trajectory of the intercepting ball given LX, LY, and θ for the client. Illustrate your method by finding the trajectory of the ball when the launcher

[2] You can look up various figures for "pixels per cm" on the internet, but they generally refer to the size of the pixels on your computer screen. Unfortunately, this information isn't relevant to our problem because we're interested in figuring out how far the ball in the video moves *in the real world* when the image of the ball moves by one pixel on our screens. This depends on the distance from the ball to the camera, the lens, the resolution of the camera, and so forth, so it's not something that you can just look up.

is located at $(\mathrm{LX}, \mathrm{LY}) = (640, 0)$, the launcher velocity $V = 75$ (pixels per frame), and the launcher angle $\theta = 5\pi/6$.

5. Finding Where the Trajectories Intersect Each Other

We now have equations for the position of the two balls giving x and y positions in terms of time. We now want to eliminate time from the problem (for now) and write the flight path of each ball in the form $y = f(x)$. As an example, suppose we had a ball whose motion was described by

$$y = -4.9t^2 + 2t + 3$$
$$x = 5t + 2$$

We could work with this system of equations to eliminate t and get an equation for y in terms of x. Do this example explicitly as a warmup.

1. Take your pair of equations $y = \mathrm{py}(t)$ and $x = \mathrm{px}(t)$ for the path of the incoming ball from above and eliminate t to get a single equation in the form $y = ax^2 + bx + c$ describing the flight path of the incoming ball. Plug in the x values from the data table on page 2 and make sure that the resulting y values of your function are close to matching the corresponding y values in the data table. Please include a table of this data in your lab.
2. Take your pair of equations $y = \mathrm{PY}(t)$ and $x = \mathrm{PX}(t)$ for the flight path of the intercepting ball and eliminate t to get another equation in the form $y = Ax^2 + Bx + C$. Use the demonstration values $(\mathrm{LX}, \mathrm{LY}) = (640, 0)$, $V = 75$ (pixels per frame), and $\theta = 5\pi/6$. As a check on your work, make sure that when $x = 640$, the resulting y is 0.
3. Find the two points in the x, y plane where the parabolas $y = ax^2 + bx + c$ (describing the flight path of the incoming ball) and $y = Ax^2 + Bx + C$ (describing the flight path of the intercepting ball) intersect. These are the possible *locations* of the interception. Pick one of them, and explain why you picked the one that you did. We'll call this point $(\mathrm{IX}, \mathrm{IY})$ for (Interception-x and Interception-Y).

6. Figuring Out When to Launch the Intercepting Ball

You have now picked a point $(\mathrm{IX}, \mathrm{IY})$ which you know is on the flight path of each of the two balls. This means that at some time, each ball will be at this location. The remaining question is how to choose a launch time for the intercepting ball which will result in both balls being at $(\mathrm{IX}, \mathrm{IY})$ at the same time.

1. Find the time t when the incoming ball arrives at $(\mathrm{IX}, \mathrm{IY})$. We'll call this time IT (for interception time).
2. Find the value of t which makes $\mathrm{PX}(t) = \mathrm{IX}$. Now we derived the $\mathrm{PX}(t)$ equation assuming that the intercepting ball was launched at time 0, so this value of t is really the *amount of time it takes the intercepting ball to travel from the cannon to the interception point* $(\mathrm{IX}, \mathrm{IY})$.

3. Use your answers to these questions to find out when to launch the intercepting ball in order to make the interception. Please explain your reasoning.
4. (Extra credit.) Go ahead do the procedure above *without* making the initial assumptions that $(\mathrm{LX}, \mathrm{LY}) = (640, 0)$, $V = 75$, and $\theta = 5\pi/6$ to get general answers for (IX, IY), IT and t_0 in terms of the variables $(\mathrm{LX}, \mathrm{LY})$, V and θ. You will probably need to use *Mathematica* or another computer system to do the algebra.

7. Grading Standards for This Lab

Basically, you will be graded on your explanation to the client of your procedures for solving this problem. The example computation is not the point of the lab, but it is the final test of your methods. If your methods (as carried out on the example) fail to work, it will cost you one letter grade.

A. The lab is clearly explained and easy to follow for any client. The procedures are "ready to code". Explanations show a deep conceptual understanding of the mathematics of the various problems. Algebra and computations are clear correct and the resulting interception solution is clear and well-supported and actually results in an interception of the example ball with the example cannon.

B. EITHER The lab is explained, but the explanation requires the client to be familiar with the problem or requires significant additional work in order for the client to implement the solution. The example computation is supported, but may be hard to follow. It still results in an interception of the example ball with the example cannon OR The explanation of the lab meets the standards for the 'A' grade, but the example solution fails to intercept the incoming ball.

C. EITHER The explanation in the lab is incomplete and/or does not show a conceptual understanding of the problem. Procedures are incompletely specified or require substantial effort to interpret. However, the example computation is supported and results in an interception. OR The explanation meets the standards for the 'B' grade, but the example solution fails to intercept the incoming ball.

D. Grades of 'D' or below are assigned to labs where the exposition is exceptionally poor and/or the example solution fails to result in an interception.

8. Group work Standards for this Lab

The rule for group work on the labs is simple:

Work together, write separately, acknowledge everything.

This means that you need to write up everything in your lab yourself *including calculations*. Of course, it is fine if your calculations reproduce the results of other members of your lab group– you are welcome to share approaches and ideas with the other students, and correct calculations following the same approach *should* lead you to the same numbers.

Similarly, it's encouraged for other students to teach you how to use *Mathematica*, but you need to have the experience of facing down the computer yourself– I don't want you to take a screenshot of somebody else's computer work and turn it in as your own.

You are welcome to complete the lab using alternative technology (such as your graphing calculator), but I encourage you to learn Mathematica or Wolfram Alpha. At this point in your college career, you're nearing the limits of the graphing calculator's capabilities, and it's time to move up.

You may turn in work prepared on the computer (using Word or something similar), handwritten work, or any mixture of the two. It's sometimes convenient to mix the two by taking photographs of your handwritten work and including them in the lab. (Of course, you can't take photographs of someone else's handwritten work and include them in the lab!)

9. Acknowledgements

I'm very much indebted to the Fall 2013 class for asking great questions and helping me refine this lab assignment. In particular, students Steven Brinkley, Madison Canning, Natalie Geng, and Salem Rustom spent a lot of time with me helping me come up with a better explanation of what the lab is about.