Math 317: Homework 8

Due Saturday, April 13, 2019

1. (29.4) Let f and g be differentiable functions on an open interval I. Suppose there are $a, b \in I$ with $a < b$ and $f(a) = f(b) = 0$. Show that there exists $x \in I$ so that

   $$f'(x) + f(x)g'(x) = 0.$$

   Hint. Consider $h(x) = f(x)e^{g(x)}$.

2. (31.2) Consider the *hyperbolic sine and cosine* functions, $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

   a. Find the Taylor series for $\sinh(x)$ and show that it converges to $\sinh(x)$.

   b. Find the Taylor series for $\cosh(x)$ and show that it converges to $\cosh(x)$.

3. (31.6) We're going to work through an alternative proof of Taylor's theorem, for Taylor series centered at $c = 0$.

   Let f be defined on (a, b) with $a < 0 < b$ and assume that, for given $n \in \mathbb{N}$, $f^{(n)}$ exists on (a, b). Let $x > 0$, and take M to be the unique number so that

   $$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}$$

   is valid. Define

   $$F(x) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M\frac{(x-t)^n}{n!}$$

   for $t \in [0, x]$.

   a. Show that F is differentiable on $[0, x]$ and that furthermore

   $$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M].$$

   b. Show $F(0) = F(x)$.

   c. Apply Rolle's Theorem to obtain $y \in (0, x)$ so that $f^{(n)}(y) = M$.

4. Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$.

   a. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$.

   Prove that (s_n) is a convergent sequence. *Hint.* To show that (s_n) is Cauchy, first show that $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$.

b. Show that f has a *fixed point.* In other words, there is a number s so that $f(s) = s$.