Math 317: Homework 11

Due Friday, May 3, 2019

1. (34.4) Let f be the function,

$$f(t) = \begin{cases} t & t < 0 \\ t^2 + 1 & 0 \leq t \leq 2 \\ 0 & t > 2 \end{cases}$$

   a. Determine the function $F(x) = \int_0^x f(t)\, dx$.

   b. Sketch F. Where is F continuous?

   c. Where is F differentiable? Calculate F' at the points of differentiability.

2. (34.6) Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_0^{\sin x} f(t)\, dt \quad \text{for } x \in \mathbb{R}.$$

   Show that F is differentiable on $\mathbb{R}$ and compute F'.

3. (34.11) Suppose f is a continuous function on $[a, b]$. Show that if $\int_a^b f(x)^2\, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

4. (34.12) Show that if f is a continuous real-valued function on $[a, b]$ satisfying $\int_a^b f(x)g(x)\, dx = 0$ for every continuous function g on $[a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.

5. For this problem, you may use the results of questions (4) and (5) freely. Let $C([a, b])$ be the set of all continuous functions on the interval $[a, b]$. Define a function

$$\langle \cdot, \cdot \rangle : C([a, b]) \times C([a, b]) \to \mathbb{R}$$

   by,

$$\langle f, g \rangle = \int_a^b fg$$

   a. Let $f, g, h \in C([a, b])$. Show that $\langle \cdot, \cdot \rangle$ is an *inner product*. That is, show each of:

      - $\langle f, g \rangle = \langle g, f \rangle$.
      - $\langle af, g \rangle = a\langle f, g \rangle$.
      - $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$.
      - $\langle f, f \rangle = 0$ if and only if $f = 0$.

b. Let $f \in C([a, b])$. Show that if $\langle f, g \rangle = 0$ for all $g \in C([a, b])$, then $f = 0$. In other words, you are showing that the only function *orthogonal* to all other functions with this inner product is the zero function.