Math 317: Homework 1

Due Friday, February 1, 2019

1. (1.1) Prove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

2. (1.6) Prove $11^n - 4^n$ is divisible by 7 when n is a positive integer. (An integer k is divisible by an integer q if there exists an integer m so that $k = mq$.)

3. (4.3, partial) For each subset of $\mathbb{R}$ below, determine both the **supremum** and the **infimum**, if they exist. If either doesn't exist, say so. You do not need to give a rigorous proof of your answer.

   a. $A = \{3, 4, 5\}$

   b. $B = \left\{ n + \dfrac{(-1)^n}{n} \ :\ n \in \mathbb{N} \right\}$

   c. $C = \left\{ \sin\left(\dfrac{n\pi}{3}\right) \ :\ n \in \mathbb{N} \right\}$

   d. $D = \{r \ :\ r \in \mathbb{Q},\ r^2 < 3\}$

   e. $E = \left\{ 1 - \dfrac{1}{3^n} \ :\ n \in \mathbb{N} \right\}$

4. Don't worry about writing out any formal proofs in this problem. Decide whether each of the following statements is true. If the statement is **true**, you don't need to do anything more. If the statement is **false**, give a concrete example (that is, a counterexample) that shows the statement failing.

   a. For a nonempty, bounded set $S \subseteq \mathbb{R}$, $\inf S < \sup S$.

   b. If $r \neq 0$ is rational and α is irrational, then $r\alpha$ is irrational.

   c. If $T \subset S \subset \mathbb{R}$, T is nonempty and S is bounded, then $\sup T \leq \sup S$.

   d. A finite, nonempty set always contains its supremum.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set $\{-x \ :\ x \in A\}$. Prove that

$$\inf A = -\sup(-A).$$

   (Note: This is the bulk of the proof of Corollary 4.5. You're welcome to make use of all of the theorems and properties in Section 3, including Theorem 3.2. $\mathbb{R}$ is an ordered field.)