Math 317 Midterm Exam #2 Practice Problems

1. Give an example for each of the following, or explain conclusively and clearly why one cannot exist, stating any facts, definitions, and theorems that apply.

   (a) A series which converges, but does not converge absolutely.

   Example. $\sum(-1)^n(1/n)$

   (b) A function which is uniformly continuous on a set S and unbounded on S.

   Example. $f(x) = x$ on the set $S = \mathbb{R}$. (On the other hand, if S is bounded, then a uniformly continuous function must be bounded on S)

   (c) A function f and a sequence (x_n) so that (x_n) converges, but $(f(x_n))$ diverges.

   Example. $f(x) = 1/x$, together with the sequence $x_n = 1/n$. Then $\lim f(x_n) = \lim n = +\infty$, so the sequence $(f(x_n))$ diverges.

   (d) A power series $g(x)$ with radius of convergence 1 that converges for $x = -1$.

   Example. $g(x) = \sum (1/n)x^n$

   (e) A function which is uniformly continuous but not continuous.

   Cannot exist. A function that is uniformly continuous is also continuous. (Uniform continuity is more restrictive than continuity)

   (f) A uniformly convergent function series that is not a power series.

   Example. $g(x) = \sum \frac{\sin x}{n^2}$ converges uniformly to a function on $\mathbb{R}$ by the Weierstrauss M test, with sequence $M_n = 1/n^2$.

   (g) A sequence (x_n) for which $\liminf x_n \neq \limsup x_n$.

   Example. A divergent sequence which does not diverge to $\pm\infty$ will suffice here. For example, $x_n = \sin n$.

   (h) A sequence (s_n) for which $\liminf s_n$ is undefined.

   Cannot exist. Both lim inf and lim sup always exist for a sequence of real numbers, although they may either be real or $\pm\infty$.

2. Answer whether each statement is true or false. If the statement is true, give a brief explanation. If the statement is false, provide a counterexample.

   (a) Any function $f : \{3\} \to \mathbb{R}$ is continuous at 3.

   True. The only sequence (x_n) in the domain of f tending to 3 is the sequence $(x_n) = 3$, and $\lim f(x_n) = \lim f(3) = f(3)$ so f is continuous by definition.

   (b) If (x_n) is a sequence of real numbers and (x_{n_k}) is any subsequence, then

   $$\limsup_{n\to\infty} x_n \leq \limsup_{k\to\infty} x_{n_k}.$$

   False. For example, consider the sequence $x_n = (-1)^n$ with subsequence $x_{n_k} = -1$.

   (c) If a function $f(x)$ is continuous on a set S and (x_n) is a sequence in S which converges to a real number x, then $\lim f(x_n) = f(x)$.

   False. If the limit x is not in S, then continuity of f on S says nothing about $f(x)$. For instance, $f(x) = 1/x$ is continuous on the set $(0, 1)$, and the sequence $x_n = 1/n$ converges to 0, but $f(x)$ does not exist and hence cannot equal $\lim f(1/n) = +\infty$.

(d) Every sequence of continuous functions converges pointwise to a continuous function.

False. Only uniform convergence of continuous functions guarantees continuity. For example, consider the function sequence $f_n(x) = \frac{x^n}{1+x^n}$ on domain $[0, 2]$.

(e) Any function $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{c\to 0} h(x + c) - h(x - c) = 0$$

is continuous.

False. Consider the function $h(x) = 0$ if $x \neq 0$ and $h(x) = 1$ if $x = 0$. Then at $x = 0$, $h(c) = h(-c) = 0$ for all $c \neq 0$, so $\lim_{c\to 0} h(c) - h(-c) = 0$. But $h(x)$ is not continuous at $x = 0$. Furthermore, at $x \neq 0$, the limit required is still 0 (why?). So h satisfies the condition for all $x \in \mathbb{R}$ but is not continuous.

(f) The equation $2x^{14} - 10x^7 + 2 = 5x^{61} + 4x^{19} - 1$ has a solution in the interval $[0, 1]$.

True. Let $f(x) = 2x^{14} - 10x^7 + 2 - 5x^{61} - 4x^{19} + 1$. Then if $f(x)$ has a zero on $[0, 1]$, the equation has a solution on $[0, 1]$. Now, $f(1) = 2-10+2-5-4+1 = -14$ while $f(0) = 2 + 1 = 3$. As f is a polynomial function, it is continuous, hence by the Mean Value Theorem, there exists a number $x \in [0, 1]$ so that $f(x) = 0$ because $-14 < 0 < 3$. So, there **is** a solution on the interval.

3. Define the series $\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$. Explain why the series converges uniformly to a continuous function $W_0(x)$ on the interval $(-1/e, 1/e)$. (The function $W_0(x)$ actually can be defined on a larger domain and is called the main branch of the *Lambert W function.* One nice property of this function is that, $W_0(xe^x) = x$ whenever $x \geq -1$.)

Solution. We have that $1/R = \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$. Now,

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{(n+1)^n}{(n+1)n!}\frac{n!}{n^{n-1}} = \limsup \left(\frac{n+1}{n}\right)^{n-1} = e.$$

So $1/R \leq e$ implies that $1/e \leq R$, so as the power series converges uniformly on $(-R, R)$ it also converges uniformly on $(-1/e, 1/e) \subseteq (-R, R)$. As each partial sum in the power series is a polynomial, it is continuous, hence its uniform limit is also continuous.

4. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} x^n$$

Solution. We compute the radius of convergence of the power series.

$$\limsup \left|\frac{2^n}{\sqrt{n}}\right|^{1/n} = \limsup \frac{2}{n^{n/2}} = 2,$$

so the radius of convergence is $1/2$. We know that the power series converges on $(-1/2, 1/2)$, but we must check the endpoints. When $x = -1/2$, the power series becomes the series

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test. On the other hand, when $x = 1/2$, the power series becomes

$$\sum \frac{1}{\sqrt{n}}$$

which diverges for example by comparison with $\sum 1/n$. Hence the interval of convergence of the power series is $[-1/2, 1/2)$.