Linear Algebra I: Homework 4

Due Friday, February 23, 2018

1. The *rotation matrix* R_θ is a matrix whose multiplication rotates vectors in $\mathbb{R}^2$ by θ radians counterclockwise. R_θ has the formula,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   a. Calculate the determinant of R_θ. Does it depend on θ?

   b. *Without doing any Gaussian elimination or using the formula for inverses of* 2×2 *matrices*, find a concise formula for $(R_\theta)^{-1}$. (Hint: Think about what R_θ does geometrically).

   c. Find a non-identity matrix $B \neq I$ for which $B^3 = BBB = I$.

2. If A is an $n \times n$ symmetric matrix (that is, $A^T = A$), and B is any $n \times m$ matrix, show that the following are symmetric:

   a. $B^T B$

   b. BB^T

   c. $B^T AB$

3. Find the following determinants:

   a. $\det \begin{pmatrix} 2 & 3 & 7 & 0 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

   b. $\det \begin{pmatrix} 1 & -4 & 1 \\ 0 & 3 & 1 \\ 2 & -8 & 2 \end{pmatrix}$

4. a. A is an $n \times n$ matrix and $\det(A) = 2$, find $\det(3A)$. (Hint: First calculate $\det(3I)$, then use that $\det(AB) = \det(A)\det(B)$.)

   b. Is det a linear function on matrices?

5. a. For an $n \times n$ matrix $A = (a^i_j)$ and a number λ, compare $\operatorname{tr}(\lambda A)$ and $\lambda \operatorname{tr}(A)$.

   b. For a pair of $n \times n$ matrices $A = (a^i_j)$ and $B = (b^i_j)$, compare $\operatorname{tr}(A + B)$ and $\operatorname{tr}(A) + \operatorname{tr}(B)$.

   c. Is tr a linear function on matrices?