Math 317 Final Exam Practice Problems

1. Give an example for each of the following, or explain conclusively and clearly why one cannot exist, stating any facts, definitions, and theorems that apply.

   (a) A bounded set which is not an interval.

   Example. The set $\{1, 2, 3\}$ is bounded, but is not an interval.

   Non-Example. The set $\mathbb{Z}$ is not an interval, but is unbounded.

   Non-Example. The set $\mathbb{N}$ is not an interval, but is unbounded.

   (b) An unbounded sequence (a_n) with $\limsup a_n = 1$.

   Example. The sequence $a_n = 1 - (1 + (-1)^n)^n$ is unbounded below, hence unbounded, but has $\limsup a_n = 1$.

   Any example for this question must not be a convergent sequence (why?) and must be bounded above. Note that 1 is not necessarily the sequence upper bound, for instance:

   Example. The sequence $a_1 = 300$, $a_n = 1 - (1 + (-1)^n)^n$ for $n \geq 2$ is bounded above by 300, but also satisfies.

   (c) A divergent sequence (s_n) with a convergent subsequence.

   Any sequence which is bounded but divergent satisfies this condition by the Bolzano-Weierstrauss theorem.

   Example. $s_n = \sin n$ is bounded by ± 1, hence satisfies.

   Any sequence with finite $\liminf$ or $\limsup$ satisfies, as the $\liminf$ and $\limsup$ are in fact the infimum and supremum, respectively, of the set of subsequential limits.

   Example. $s_n = |n \sin(n\pi/6)|$ is unbounded above, but has $\liminf s_n = 0$. So, there must be some subsequential limits which are close to 0. This means s_n must have a convergent subsequence.

   Non-Example. $c_n = n$ diverges, and both $\liminf c_n = \limsup c_n = +\infty$. We can observe that c_n has no convergent subsequence.

   Non-Example. $d_n = (-1)^n n$ diverges, and $\liminf d_n = -\infty$ while $\limsup d_n = +\infty$. We can see that d_n has no convergent subsequence.

   (d) A continuous function on $[0, 1]$ which is not uniformly continuous.

   Cannot exist. Any function which is continuous on $[0, 1]$ is uniformly continuous.

   In fact, Any function which can be *extended to* a continuous function on a closed interval $[a, b]$ is uniformly continuous.

   (e) A continuous function on $(0, 1)$ which is not uniformly continuous.

   Example. $f(x) = 1/x$

   Example. $f(x) = \sin(1/x)$

   Non-Example. $f(x) = x^2$, as this function can be extended to a continuous function on $[0, 1]$ by setting $f(0) = 0$ and $f(1) = 1$.

   (f) A sequence of functions which are uniformly continuous on $[3, 4]$ that does not converge uniformly.

   Example. $f_n(x) = (x - 3)^n$

(g) A function defined on $(0,1)$ that is not differentiable at any point in its domain.

Example. The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap (0,1) \\ 0 & x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

is not continuous anywhere, and hence is not differentiable anywhere.

Example. If $a \in (0,1)$, $b \in \mathbb{N}$ is odd, and $ab > 1 + \frac{3}{2}\pi$, the function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

called the *Weierstrauss function* is continuous everywhere but differentiable nowhere.

Example. Consider the floor function $\lfloor y \rfloor$, which is the smallest integer smaller than or equal to y. Let

$$f_n(x) = \frac{1}{2^n} \left| 2^n x - \left\lfloor 2^n x + \frac{1}{2} \right\rfloor \right|.$$

Then by the Weierstrauss M-test, we can see that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function, call that function $G(x)$. As each finite term $g_N(x) = \sum_{n=1}^{N} f_n(x)$ is continuous and $G(x)$ is the uniform limit, $G(x)$ is also continuous. But it can be shown that $G(x)$ is differentiable nowhere. (See example 38.1 in the textbook for more information)

(h) A differentiable function defined on $(0,1)$ which is not uniformly continuous.

Example. $f(x) = 1/x$

Example. $f(x) = \sin(1/x)$

(i) A bounded function on $[0,1]$ that is not integrable.

Example. The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

is not integrable, as its upper integral is 1, while its lower integral is 0.

(j) **Devise your own part to this question, and challenge your study group!**

2. Answer whether each statement is true or false. If the statement is true, give a brief explanation. If the statement is false, provide a counterexample.

(a) The set $S = \{\frac{p}{q} : p, q \in \mathbb{Z}, q > 20\}$ is bounded below.

False. Consider any integer $M < 0$. Then with $p = q(M-1) \in \mathbb{Z}, q = 21$, we have

$$\frac{p}{q} = (M-1) < M,$$

while $\frac{p}{q} \in S$.

(b) The sequence $(\frac{n}{3^n})$ is a convergent sequence.

True. The limit is 0. Try proving this by the definition of a limit. Then try proving that the sequence is bounded and Cauchy, hence convergent, without using what the limit is. Finally, try using the Monotone Convergence Theorem.

(c) The equation $\cos(x) = \tan(x)$ has a solution in $[0, \pi/4]$ (note: $\frac{1}{\sqrt{2}} < 1$).

True. Consider $F(x) = \cos(x) - \tan(x)$ and apply the Intermediate Value Theorem to show that $F(x)$ has a zero on $[0, \pi/4]$.

(d) If a function f has a maximum at $c \in \mathbb{R}$, then f is differentiable at c.

False. f might also be not differentiable. For instance, $-|x|$ is not differentiable at its maximum where $x = 0$.

(e) Suppose f and g are differentiable on all of $\mathbb{R}$. Then,

$$h(x) = (f(x))^2 - 3g(f(x))$$

is also differentiable on all of $\mathbb{R}$.

True. By the product rule, scalar multiple rule, subtraction rule, and chain rule. In fact,

$$h'(x) = 2(f(x))f'(x) - 3g'(f(x))f'(x).$$

(f) Suppose g is integrable on $[a, b]$ and that there exists $c \in (a, b)$ such that

$$\int_a^c g > \int_a^b g.$$

Then there exists $d \in (a, b)$ so that $g(d) < 0$.

True. By Theorem 33.6, $\int_a^c g + \int_c^b g = \int_a^b g < \int_a^c g$, so we see that $\int_c^b g < 0$. Now if $g(x) \geq 0$ for all $x \in [c, b]$, then by Theorem 33.4(i), $\int_c^b \geq 0$. But this is a contradiction! So there exists $x \in [c, b]$ so that $g(x) < 0$.

If $x \neq b$, we're done, as then $x \in (a, b)$. But if $x = b$, notice that we could consider the function $f(x)$ which is the same as $g(x)$ except $f(b) = 0$. Now all integrals $\int_r^s g = \int_r^s f$ as they differ only at one point. Application of the above argument shows that there exists $d \in [c, b]$ so that $f(d) < 0$. But now we know that $d \neq b$, as $f(b) = 0$. So $d \in [c, b)$. Then $g(d) = f(d) < 0$.

(g) If $h_n(x) = x - x^n$ then h_n converges uniformly on $[0, 1]$.

False. We have that the pointwise limit is,

$$h(x) = \begin{cases} 0 & x = 1 \\ 1 & x \neq 1 \end{cases}$$

The pointwise limit is the only candidate for the uniform limit. However, $h(x)$ is not continuous, while all $h_n(x)$ are. As the uniform limit of continuous functions must be continuous, and $h(x)$ is not, it must not be the uniform limit of $h_n(x)$. So we conclude no uniform limit exists.

(h) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$.
True. For all n we have

$$|a_n \sin(nx) = |a_n||\sin(n_x)| < |a_n|,$$

as $|\sin(y)| < 1$ for all y. Then as $\sum |a_n|$ converges, the Weierstrauss M-test implies that the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$.

(i) Every power series converges on some interval (a, b) with $a \neq b$.
False. $\sum_{n=1}^{\infty} n!x^n$ has radius of convergence 0 (by the ratio test), so it only converges at $x = 0$. But $\{0\}$ is not an interval of the required form.

(j) **Devise your own part to this question, and challenge your study group!**

3. Let $f(x) = |x|^3$. Compute $f^{(k)}(x)$ for all $k \geq 1$ and all $x \in \mathbb{R}$.

Answer. $f'(x) = 3x|x|$ and $f''(x) = 6|x|$. $f'''(x) = -6$ if $x < 0$ and $f'''(x) = 6$ if $x > 0$, and is undefined if $x = 0$. $f^{(k)}(x) = 0$ except at 0 where it is undefined, for $k \geq 4$.

When computing the derivatives at $x = 0$, you should use the limit definition of the derivative.

4. Use the definition of differentiability to show that $f(x) = x^2 - 1$ is differentiable at all of $\mathbb{R}$.

Answer.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - 1 - a^2 + 1}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} = \lim_{x \to a} x + a = 2a$$

5. Use the definition of integrability to show that $f(x) = 2x$ is integrable on $[0, 1]$.

Answer. Consider the partitions $P_n = \left\{0, \frac{1}{n}, \cdots, \frac{k}{n}, \cdots, 1\right\}$. Then

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} M\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right)\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{2k}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \frac{2}{n^2}\sum_{k=1}^{n} k = \frac{2}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{n}. \end{aligned}$$

A similar calculation shows that $L(f, P_n) = \frac{n-1}{n}$. Now, we know that

$$\frac{n+1}{n} = U(f, P_n) \geq U(f) \geq L(f) \geq L(f, P_n) = \frac{n-1}{n}.$$

Taking limits as $n \to \infty$ yields,

$$1 \geq U(f) \geq L(f) \geq 1,$$

so $U(f) = L(f) = 1$ and $2x$ is integrable over $[0, 1]$.

Challenge. Use the Cauchy definition of integrability instead!

6. Let $f_n(x) = \sin(x/n)$.
   (a) Find the pointwise limit f of the sequence (f_n).
   Answer. $f(x) = 0$
   (b) Does (f_n) converge uniformly to its pointwise limit f on $[-\pi, \pi]$? On all of $\mathbb{R}$?
   Answer. On $[-\pi, \pi]$, the sequence converges uniformly. To see this, notice that when $n > 2$, we have that for all $x \in [-\pi, \pi]$,
   $$|\sin(x/n)| < |\sin(\pi/n)|$$
   Now, $\lim_{n\to\infty} |\sin(\pi/n)| = 0$, so for all $\epsilon > 0$ we have N so that $n \geq N$ implies that
   $$|\sin(x/n)| < |\sin(\pi/n)| < \epsilon.$$
   Answer. On $\mathbb{R}$, the sequence does not converge uniformly. Whenever $x = n\pi/2$ we have that $\sin(x/n) = \sin(\pi/2) = 1$. For $0 < \epsilon < 1$, this means that at $x = n\pi/2$, $|\sin(x/n)| > \epsilon$.

7. Use that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ to compute the limit of the sequence $(s_n) = (n \sin(\pi/n))$.
   Answer. That $\lim_{x\to 0} \frac{\sin x}{x} = 1$ means that for all sequences $x_n \to 0$, $\lim_{n\to\infty} \frac{\sin x_n}{x_n} = 1$. So as $\pi/n \to 0$ we have that,
   $$\lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} = \pi \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = \pi.$$

8. Determine whether each of the following sequences and series converge. (Also, identify clearly which is a **sequence** and which is a **series**).
   (a) $(1 - \frac{2}{n^2})$
   Answer. This **sequence** converges to 1.
   (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$
   Answer. This **series** diverges by the comparison test: $1/\sqrt{k} > 1/k$ but $\sum 1/k$ does not converge.
   (c) $(1 + (-1)^n)^{1/n}$
   Answer. This **sequence** does not converge. Its $\liminf$ is 0, while its $\limsup$ is 1.
   (d) $\sum_{k=0}^{\infty} \frac{2^k}{k!}$
   Answer. This **series** converges by the ratio test.

9. Determine whether or not the function sequence or series converges (pointwise or uniformly) on the given domain.
   (a) The sequence (g_n) on $(0, 1)$, where $g_n(x) = \frac{x}{nx+1}$
   Answer. This sequence converges uniformly to $g(x) = 0$. We have that $0 < \frac{x}{nx+1} < \frac{1}{n}$ for all n and all $x > 0$. So when $n > N = \frac{1}{\epsilon}$,
   $$\left| \frac{x}{nx+1} - 0 \right| < \frac{1}{n} < \epsilon.$$

(b) The series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$, where $f_n(x) = 0$ if $x \leq n$ and $f_n(x) = (-1)^n$ if $x > n$.

Answer. This sequence converges pointwise to its limit, as for all x, the sum $\sum_{n=1}^{\infty} f_n(x)$ only has a finite number of nonzero terms.

The sequence does not converge uniformly. Consider $\epsilon < 1$. For $m \in \mathbb{N}$, notice that $|\sum_{n=1}^{m} f_n(2m + (1/2))| = |-1| > \epsilon$. So there cannot exist sufficient N for the definition of uniform convergence.

10. Determine the interval of convergence of the following power series.

(a) $\sum n^2 x^n$

Answer. $R = 1$, as $\limsup (n^2)^{1/n} = 1$, so the series converges on $(-1, 1)$.

As $(\pm 1)^n n^2$ does not converge to 0 as $n \to \infty$, we conclude that the series does not converge at ± 1.

So the interval of convergence is $(-1, 1)$.

(b) $\sum (x/n)^n$

Answer. $R = +\infty$, so the series converges for all x.

(c) $\sum x^{n!}$

Answer. $R = 1$, as $\limsup (a_n)^{1/n} = 1$ ($a_1 = 2$, $a_n = 1$ if $n = k!$ for some integer $k > 1$, and $a_n = 0$ otherwise). so the series converges on $(-1, 1)$.

As $(\pm 1)^n a_n$ diverges as $n \to \infty$, we conclude that the series does not converge at ± 1.

So the interval of convergence is $(-1, 1)$.

(d) $\sum \left(\frac{3^n}{n 4^n}\right) x^n$

Answer. $R = 4/3$. So, the series definitely converges on $(-4/3, 4/3)$.

We check $x = -4/3$: $\sum \left(\frac{3^n}{n 4^n}\right) (-4/3)^n = \sum \frac{(-1)^n}{n}$ which does converge by the alternating series test.

We check $x = 4/3$: $\sum \left(\frac{3^n}{n 4^n}\right) (4/3)^n = \sum \frac{1}{n}$ which does not converge.

So, the series converges on $[-4/3, 4/3)$