Math 317 Final Exam Practice Problems

1. Give an example for each of the following, or explain conclusively and clearly why one cannot exist, stating any facts, definitions, and theorems that apply.
   (a) A bounded set which is not an interval.
   (b) An unbounded sequence (a_n) with $\limsup a_n = 1$.
   (c) A divergent sequence (s_n) with a convergent subsequence.
   (d) A continuous function on $[0, 1]$ which is not uniformly continuous.
   (e) A continuous function on $(0, 1)$ which is not uniformly continuous.
   (f) A sequence of functions which are uniformly continuous on $[3, 4]$ that does not converge uniformly.
   (g) A function defined on $(0, 1)$ that is not differentiable at any point in its domain.
   (h) A differentiable function defined on $(0, 1)$ which is not uniformly continuous.
   (i) A bounded function on $[0, 1]$ that is not integrable.
   (j) **Devise your own part to this question, and challenge your study group!**

2. Answer whether each statement is true or false. If the statement is true, give a brief explanation. If the statement is false, provide a counterexample.
   (a) The set $S = \{\frac{p}{q} : p, q \in \mathbb{Z}, q > 20\}$ is bounded below.
   (b) The sequence $(\frac{n}{3^n})$ is a convergent sequence.
   (c) The equation $\cos(x) = \tan(x)$ has a solution in $[0, \pi/4]$ (note: $\frac{1}{\sqrt{2}} < 1$).
   (d) If a function f has a maximum at $c \in \mathbb{R}$, then f is differentiable at c.
   (e) Suppose f and g are differentiable on all of $\mathbb{R}$. Then,

   $$h(x) = (f(x))^2 - 3g(f(x))$$

   is also differentiable on all of $\mathbb{R}$.
   (f) Suppose g is integrable on $[a, b]$ and that there exists $c \in (a, b)$ such that

   $$\int_a^c g > \int_a^b g.$$

   Then there exists $d \in (a, b)$ so that $g(d) < 0$.
   (g) If $h_n(x) = x - x^n$ then h_n converges uniformly on $[0, 1]$.
   (h) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$.
   (i) Every power series converges on some interval (a, b) with $a \neq b$.
   (j) **Devise your own part to this question, and challenge your study group!**

3. Let $f(x) = |x|^3$. Compute $f^{(k)}(x)$ for all $k \geq 1$ and all $x \in \mathbb{R}$.

4. Use the definition of differentiability to show that $f(x) = x^2 - 1$ is differentiable at all of $\mathbb{R}$.

5. Use the definition of integrability to show that $f(x) = 2x$ is integrable on $[0, 1]$.

6. Let $f_n(x) = \sin(x/n)$.

   (a) Find the pointwise limit f of the sequence (f_n).

   (b) Does (f_n) converge uniformly to its pointwise limit f on $[-\pi, \pi]$? On all of $\mathbb{R}$?

7. Use that $\lim_{x\to\infty} \dfrac{\sin x}{x} = 1$ to compute the limit of the sequence $(s_n) = (n \sin(\pi/n))$.

8. Determine whether each of the following sequences and series converge. (Also, identify clearly which is a **sequence** and which is a **series**).

   (a) $(1 - \frac{2}{n^2})$

   (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

   (c) $(1 + (-1)^n)^{1/n}$

   (d) $\sum_{k=0}^{\infty} \frac{2^k}{k!}$

9. Determine whether or not the function sequence or series converges (pointwise or uniformly) on the given domain.

   (a) The sequence (g_n) on $(0, 1)$, where $g_n(x) = \frac{x}{nx+1}$

   (b) The series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$, where $f_n(x) = 0$ if $x \leq n$ and $f_n(x) = (-1)^n$ if $x > n$.

10. Determine the interval of convergence of the following power series.

    (a) $\sum n^2 x^n$

    (b) $\sum (x/n)^n$

    (c) $\sum x^{n!}$

    (d) $\sum \left(\frac{3^n}{n4^n}\right) x^n$