Math 317 Midterm Exam #1 Practice Problems

1. Give an example for each of the following, or explain conclusively and clearly why one cannot exist, stating any facts, definitions, and theorems that apply.

   (a) An alternating sequence which converges. (A sequence is *alternating* if its terms alternate between positive and negative)

   Example. $a_n = (-1)^n/n$.

   (b) A sequence with no convergent subsequence.

   Example. $b_n = n$. If the sequence had to also be **bounded**, then no convergent subsequence would exist (Bolzano-Weierstrauss)

   (c) A decreasing sequence which diverges.

   Example. $c_n = -n$. If the sequence had to also be bounded, then it would have to converge (Monotone Convergence Theorem for sequences)

   (d) A sequence (s_n) with $\liminf s_n = -\infty$ and $\limsup s_n = +\infty$.

   Example. $s_n = (-n)^n$.

2. Answer whether each statement is true or false. If the statement is true, give a brief explanation. If the statement is false, provide a counterexample.

   (a) Every bounded sequence contains its supremum.

   False. For example, the sequence $a_n = 1 - 1/n$ does not contain its supremum 1.

   (b) Every Cauchy sequence is bounded.

   True. That every Cauchy sequence is bounded is a lemma we discussed in class. Similarly, you could reason that every Cauchy sequence converges, and that every convergent sequence must be bounded.

   (c) Every increasing sequence is bounded.

   False. The sequence $c_n = n$ is increasing but unbounded.

   (d) If a sequence converges to a number L, then so do all subsequences.

   True. This is a theorem about convergent sequences.

3. Find the limit of the sequence $\left(\dfrac{2n-3}{3n+1}\right)$ and prove that your limit is correct.

   Proof. We claim that the limit is $2/3$. Let $\varepsilon > 0$ and let

   $$N = \frac{1}{3}\left(\frac{11}{3\varepsilon} - 1\right).$$

   Then we have that if $n > N$,

   $$\left|\frac{2n-3}{3n+1} - \frac{2}{3}\right| = \frac{11}{3(3n+1)} < \frac{11}{3(3N+1)} = \varepsilon,$$

   showing that the sequence indeed converges to $2/3$. $\square$

4. Prove that the sequence (s_n) defined by $s_1 = 1$ and $s_{n+1} = \sqrt{3 + s_n}$ does **not** converge to 4.

   Warning. It is **not sufficient** to show that the sequence is bounded above by 4!

   Proof. We claim that the sequence is bounded by 3. Certainly $s_1 = 1 < 3$. Now, suppose $s_k < 3$. Then $3 + s_k < 3 + 3$ and so $\sqrt{3 + s_k} < \sqrt{6} < 3$, so $s_{k+1} < 3$. By induction, we conclude that $s_n < 3$ for all n. So $\sup\{s_n : n \in \mathbb{N}\} \leq 3$ and as $\lim s_n$ (if it exists) must be less than $\sup\{s_n : n \in \mathbb{N}\}$, (s_n) cannot converge to 4. $\square$

5. A sequence is *periodic* if there exists $p \in \mathbb{N}$ so that $a_{n+p} = a_n$ for all natural numbers n. Let a_n be a periodic sequence for which $a_1 \neq a_{2019}$. Prove that (a_n) does not converge. (Hint: Is it Cauchy?)

   Proof. As $a_1 \neq a_{2019}$, $|a_1 - a_{2019}| > 0$. Let N be any real number, and consider

$$0 < \varepsilon < |a_1 - a_{2019}|.$$

   The sequence is periodic, so there is a number $p \in \mathbb{N}$ (called the period) so that $a_{n+p} = a_n$ for all natural numbers n.

   By induction, $a_{\ell+kp} = a_\ell$ for all natural numbers ℓ, k (Proof: $a_{\ell+p} = a_\ell$ and if we assume that $a_{\ell+kp} = a_\ell$ then $a_{\ell+(k+1)p} = a_{(\ell+kp)+p} = a_{\ell+kp} = a_\ell$). By the Archimedean principle, there exists a natural number k so that $kp > N - 1$, so $1 + kp > N$.

   But then $|a_{1+kp} - a_{2019+kp}| = |a_1 - a_{2019}| > \varepsilon$ even though $n = 1 + kp$ and $m = 2019 + kp$ are both larger than N. So a_n cannot be Cauchy as N was arbitrary, and so a_n cannot converge. $\square$