Math 301 Final Exam Practice Problems

1. Your answer to the following questions does not need to contain any explanation. If a number is naturally written using binomials, factorials, or powers, then please write it that way.

   Practice note: You should explain every part of your answer as best you can.

   (a) What is the sum of the coefficients in the expansion of $(x + y + z)^{16}$?

   Solution: The sum of the coefficients is found when $x = y = z = 1$. Equivalently, the sum of the coefficients is the number of sequences of length 16 chosen from the alphabet $\{x, y, z\}$. In either case, we see the answer is:

   $$3^{16}$$

   (b) A 2-colorable graph has 18 vertices. What is the largest number of edges the graph can have?

   Solution: We consider the complete bipartite graphs of the form $K_{j,18-j}$ which each have $j(18 - j)$ edges. The maximum is attained with $j = 9$ in $K_{9,9}$:

   $$81$$

   (c) What is the coefficient of x^7 in the expansion of $(2 + x)^{26}$?

   Solution: The term would be $\binom{26}{7}x^7y^{26-7}$ in the expansion of $(y + x)^{26}$, but we've substituted $y = 2$ and find the coefficient:

   $$2^{26-7}\binom{26}{7}$$

   (d) How large does N have to be to guarantee that any list of N integers contains at least 4 integers which have the same remainder when divided by 19?

   Solution: By the Generalized Pigeonhole Principle, the answer is

   $$19(4 - 1) + 1 = 58$$

   (e) How many subsets with 5 elements does a set with n elements have? Your answer should be in terms of n.

Solution: This is exactly a binomial question;

$$\binom{n}{5}$$

(f) Use that $16 \times 10 = 53 \times 3 + 1$ to solve for $0 \leq x < 53$ satisfying:

$$16x \equiv 6 \mod 53$$

Solution: The fact says that 10 is the multiplicative inverse of 16 modulo 53, so we multiply both sides by 10 to solve;

$$x \equiv 60 \mod 53,$$

and we subtract off 53 to yield the smallest positive x value:

$$x = 7$$

(To check our answer, notice that $16x = 16(7) = 112 = 53 \times 2 + 6$)

(g) How many ways are there to distribute 7 identical nickels and 23 identical quarters to 6 of your friends if everyone has to get at least 25 cents?

Solution: One way for everyone to get 25 cents is if everyone gets at least one quarter:

$$\binom{23-1}{6-1} \times \binom{7+6-1}{6-1}$$

Another way, is that perhaps one person gets no quarters, but instead gets 5 nickels, totaling 25 cents.

$$6 \times \binom{23-1}{5-1} \times \binom{2+6-1}{6-1}$$

Both of these cases account for all possibilities, and both are exclusive. So, the final answer is their sum:

$$\binom{23-1}{6-1} \times \binom{7+6-1}{6-1} + 6 \times \binom{23-1}{5-1} \times \binom{2+6-1}{6-1}$$

(h) How many ways are there to distribute 200 identical nickels and 40 identical quarters to 6 of your friends if everyone has to get exactly 75 cents?

Solution: There are enough quarters for everyone to get 3 quarters, and enough nickels for everyone to get 15 nickels so the question becomes, "each of your friends gets between 0 and 3 quarters and is topped off to 75 cents with nickels; how many ways are there for this to happen?"

So, we're simply picking a sequence of 6 letters from the alphabet $\{0, 1, 2, 3\}$;

$$4^6$$

(i) A planar map has 10 vertices and 20 edges. How many faces does it have?

Solution: By Euler's theorem, we know that $10 - 20 + F = 2$, and we can solve $F = 12$

(j) How many anagrams are there of the word ASSASSINATION so that no two vowels are next to each other?

Solution: The vowels are AAAIIO; there are 6 in total: 3A, 2I, 1O. The remaining consonants are SSSSNNT; there are 7 in total: 4S, 2N, 1T. There are thus

$$\binom{7}{4; 2; 1}$$

consonant anagrams, then 9 places to put 6 non-adjacent vowels for

$$\binom{9}{6}$$

vowel positions, and then

$$\binom{6}{3; 2; 1}$$

vowel anagrams. We multiply these together (we are doing these steps in sequence) to get,

$$\binom{7}{4; 2; 1}\binom{9}{6}\binom{6}{3; 2; 1}$$

Bonus Q: What about the (poorly censored) word BUTTBUTTINATION? There are BBTTTTTNN consonants and UUIIAO vowels and we get the answer

$$\binom{9}{2; 5; 2}\binom{11}{6}\binom{6}{2; 2; 1; 1}$$

(k) How many ways are there to seat (indistinguishable) people in a row of 6 chairs so that no two people have to sit next to each other?

Solution: This was on our first exam; the solution X_n satisfies a Fibonacci recurrence (why?). So with base case $X_0 = 1$ and $X_1 = 1$ we have and recurrence $X_n = X_{n-1} + X_{n-2}$ we get the 7th Fibonacci number;

$$X_6 = F_7 = 13$$

(l) How many ways are there to tile a $2 \times n$ chessboard with 2×1 dominoes? Hint: Use a recurrence relation.

Solution: The number of fillings R_k of a $2 \times k$ chessboard satisfies a Fibonacci recurrence: $R_k = R_{k-1} + R_{k-2}$ with base cases $R_1 = 1$ and $R_2 = 2$. So the answer is the $(n+1)$st Fibonacci number:

$$R_n = F_{n+1}$$

(m) How many edges does a tree with 14 vertices have?

Solution: There is one fewer edge than vertices in a tree, so

$$13$$

2. Use induction to prove that $3n + 2 \leq n^2$ for all integers $n \geq 4$.

Solution: Base case; $12 + 2 \leq 4^2$.

Inductive hypothesis; assume it's true for some $n \geq 4$; consider

$$3(n+1) + 2 = (3n+3) + 2 \leq n^2 + 2 = n^2 + 1 + 1 \leq n^2 + 2n + 1 = (n+1)^2$$

So by the principle of induction, the fact is true.

3. Let F_n denote the nth Fibonacci number, starting with $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and then defined by the recurrence $F_k = F_{k-1} + F_{k-2}$.

Use induction to prove that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

Solution: Base case; $F_1 = 1 = 2 - 1 = F_3 - 1$

Inductive hypothesis; assume it's true for some n. Consider $n + 1$:

$$F_1 + F_2 + \cdots + F_n + F_{n+1} = (F_{n+2} - 1) + F_{n+1} = F_{n+3} - 1$$

So by the principle of induction, the fact is true.

4. (a) Draw the tree T corresponding to the Prüfer code 2942701234.

   (b) How many edges does T have?

   Solution: 11 edges, 12 vertices.

   (c) How many faces are there in any planar drawing of the tree T?

   Solution: Any tree always has 1 face in a planar drawing by Euler's theorem.

5. (6 points) For a vertex v in a graph, remember that $\deg(v)$ is its degree.
   If the graph G has n vertices $V = \{v_1, v_2, \cdots, v_n\}$, prove that the quantity

$$n \deg(v_1) \deg(v_2) \cdots \deg(v_n)$$

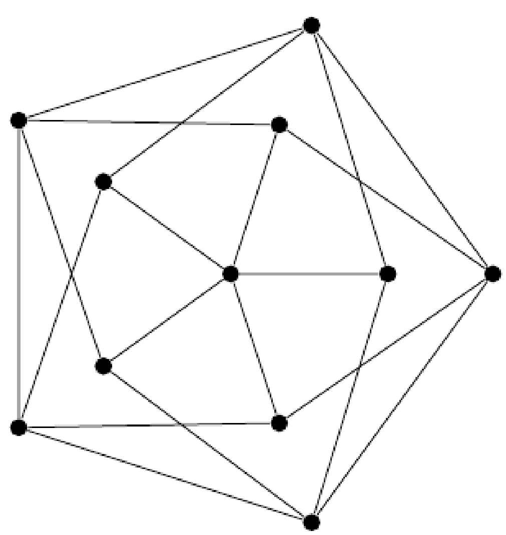
   is even.

   Solution: We know that the sum of vertex degrees is twice the number of edges, and hence an even number:

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

   If the quantity under consideration were odd, then the LHS would be a sum of an odd number of odd quantities, and hence would be odd; but this is a contradiction. □

6. The following questions are about the graph G drawn below, which has 11 vertices and 20 edges.



(a) Is G 2-colorable? Explain.

Solution: No; the outer pentagon is an odd cycle.

(b) For which number k does Brooks's theorem guarantee: You can definitely color G with k-colors.

Solution: The maximum vertex degree in G is 5 (the central vertex), so we can color the graph with 6 colors by Brooks's theorem.

(c) Show by example that you can 4-color G.

(d) Draw a spanning tree of G.

Solution: A spanning tree will have 10 edges.

(e) Does G have any Eulerian walks? Explain.

Solution: No—there are more than 2 vertices with an odd degree.

(f) G has a Hamiltonian cycle. Can you find it?

Solution: *Finding* a Hamiltonian cycle is a poor exam question, but knowing what one is is important!