

Math 301 Midterm Exam #2 Practice Problems Solutions

1. Your answer to the following questions can just be a number. If your number is naturally written as a product, using for instance factorials or binomial coefficients, please leave it that way!

You should explain your reasoning as best you can: Correct answers will receive full credit, but I can only award partial credit if you show your process.

Practice note: You should explain every part of your answer as best you can.

- (a) Find an integer x between 0 and 42 that satisfies $4x \equiv 1 \pmod{43}$.
11
- (b) Find an integer x between 0 and 30 that satisfies $7x \equiv 10 \pmod{31}$.
28
- (c) Find an integer x between 0 and 46 that satisfies $8x \equiv 4 \pmod{47}$.
24
- (d) How many graphs have *exactly* 8 (labeled) vertices $\{a, b, c, d, e, f, g, h\}$?
 2^{28}
- (e) How many subgraphs of K_8 (with vertices labeled $\{a, b, c, d, e, f, g, h\}$) have exactly 8 (labeled) vertices?
 2^{28}
- (f) How many walks of 5 steps are there in K_8 ?
 $8(7^5)$
- (g) How many *closed* walks of 5 steps are there in K_8 ? Say that two closed walks are definitely different if they have the same start/end vertex.
 $8(7^4)1$
- (h) How many walks of 5 steps are there in C_8 ?
 $8(2^5)$
- (i) How many *closed* walks of 5 steps are there in C_8 ? Say that two closed walks are definitely different if they have the same start/end vertex.
0
- (j) How many *closed* walks of 6 steps are there in C_8 ? Say that two closed walks are definitely different if they have the same start/end vertex.
 $8\binom{6}{3}$
- (k) How many subgraphs does C_3 have? Say the vertices are labeled $\{a, b, c\}$.
18
- (l) How many subgraphs does P_3 have? Say the vertices are labeled $\{a, b, c, d\}$.
34
2. Answer whether each statement is “True” or “False”. No justification is needed. No partial credit.

Practice note: You should try proving each statement and justifying your answer during practice, even though this type of question on an exam would not require it.

- (a) The complement of P_6 , the path graph on 7 vertices, has an Eulerian walk.
True.
- (b) The complement of C_6 , the cycle graph on 6 vertices, has an Eulerian walk.
False.
- (c) The complement of a disconnected graph of at least 4 vertices is always connected.
True.
- (d) The complement of a connected graph of at least 4 vertices is always disconnected.
False.
- (e) A graph with n vertices always has at least 2^n subgraphs.
True.
- (f) A graph with n edges always has at least 2^n subgraphs.
True.
- (g) For every pair of positive integers a, b there exist integers m, n such that $1 = ma + nb$.
False.
- (h) For every pair of positive integers a, b there exist integers m, n such that $\gcd(a, b) = ma + nb$.
True.
- (i) There exists a graph of 7 vertices of total vertex degree 44.
False.
- (j) For every positive integer m and integer $1 \leq a < m$ there exists an integer x so that $ax \equiv 1 \pmod{m}$.
False.
- (k) There exists a graph of 7 vertices of degrees 2, 2, 2, 2, 2, 3, 7.
False.
- (l) There exists a graph of 7 vertices of degrees 2, 2, 2, 2, 3, 5, 7.
False.
- (m) There exists a connected graph of 7 vertices of degrees 1, 1, 1, 1, 2, 2, 2.
False.
- (n) K_5 contains a closed Eulerian walk.
True.
- (o) K_6 contains a closed Eulerian walk.
False.
- (p) Every graph with all even vertex degrees has a Hamiltonian cycle.
False.
- (q) If $a \nmid b$ and $b \mid c$ then $a \nmid c$.
False.
- (r) If $a \nmid b$ and $b \nmid c$ then $a \nmid c$.
False.
- (s) If $a \equiv b \pmod{c}$ and $b \equiv c \pmod{a}$ then $a \equiv c \pmod{b}$.
False.

(t) If $a \equiv 0 \pmod{b}$ and $b \equiv 0 \pmod{c}$ then $a \equiv 0 \pmod{c}$.

True.

(u) There exists a graph of 7 vertices of degrees 2, 2, 2, 2, 3, 3, 6.

True.

(v) There exists a graph of 7 vertices of degrees 2, 2, 2, 2, 2, 5, 6.

False.

3. Use the fact that a cycle-free graph on n vertices has at most $n - 1$ edges to prove that: If G is a connected graph with at least 5 vertices and no cycles, then its complement \bar{G} has at least one cycle.

Proof. Let e, \bar{e} be the number of edges in G and \bar{G} respectively. As \bar{G} is the complement of G we know that $\bar{e} = \binom{n}{2} - e$. If G has no cycles, then $e \leq n - 1$. If in addition \bar{G} also has no cycles, then $\bar{e} \leq n - 1$. But notice then that

$$n - 1 \geq \bar{e} = \binom{n}{2} - e \geq \binom{n}{2} - (n - 1) = (n - 1)(n/2 - 1)$$

As $n - 1 > 0$ this can only happen if

$$1 \geq n/2 - 1 \Leftrightarrow 4 \geq n.$$

But we said that $n > 5$, so this is a contradiction, and so \bar{G} must be cycle-free.

4. (a) Draw a connected graph on 8 vertices for which removing any edge makes the graph disconnected.
- (b) Draw a connected graph on 8 vertices for which removing any edge leaves the graph connected.
- (c) Draw a connected graph on 8 vertices for which removing any *two* edges leaves the graph connected.
- (d) Draw a connected graph on 8 vertices that has no cycles and *is not the path graph*.